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PARAMETRIC RESONANCE IN THE PROBLEM OF A HEAVY RIGID BODY ROLLING ALONG A STRAIGHT LINE ON A PLANE†

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The problem of the stability of a heavy rigid body, bounded by the surface of an ellipsoid and with a cavity in the form of a coaxial ellipsoid, rolling along a straight line on a horizontal rough plane is investigated. It is shown that in the case of a body that is close to being dynamically symmetrical, parametric resonance always occurs leading to instability of the rolling. Each ellipsoid has its own "individual" resonance angular velocity. In the general case, regions in which the necessary stability conditions are satisfied can be distinguished in parameter space. The problem of calculating the resonance coefficient corresponding to instability for parametric resonance in a reversible third-order system is solved. © 2003 Elsevier Science Ltd. All rights reserved.

1. PARAMETRIC RESONANCE IN A REVERSIBLE THIRD-ORDER SYSTEM

Consider a quasi-autonomous reversible third-order system of differential equations, 2π-periodic with respect to t,

$$\dot{u}_s = U_s^0(u,v) + \varepsilon U_s(\varepsilon, u, v, t), \quad s = 1, 2, \quad \dot{v} = V^0(u,v) + \varepsilon V(\varepsilon, u, v, t) \tag{1.1}$$

(ε is a small parameter), invariant under the replacement $(u_1, u_2, v, t) \to (u_1, u_2, -v, -t)$. We will assume that the generating system obtained from (1.1) when $\varepsilon = 0$ allows of a constant solution $(u_1^0, u_2^0, 0)$, belonging to a fixed set $\mathbf{M} = \{u_1, u_2, v\colon v = 0\}$. The equations in variations, set up for this solution, have the form

$$\dot{x}_s = b_s^0 y, \quad s = 1, 2, \quad \dot{y} = a_1^0 x_1 + a_2^0 x_2 \tag{1.2}$$

(a_s^0, b_s^0 are constants). If $a_1^0 b_1^0 + a_2^0 b_2^0 \neq -k^2, k \in \mathrm{N}$ or $a_1^0 b_1^0 + a_2^0 b_2^0 = 0$, but $|a_1^0| + |a_0^2| \neq 0$, system (1.1), for fairly small $\varepsilon \neq 0$ allows of the 2π-periodic solution [1]

$$u_s^*(\varepsilon, t) = u_s^0 + \varepsilon u_s^1(\varepsilon, t), \quad s = 1, 2, \quad v^*(\varepsilon, t) = \varepsilon v_1(\varepsilon, t) \tag{1.3}$$

which is symmetrical with respect to the set $\mathbf{M}$. We will assume that the function $u_s^1(\varepsilon, t)$ and $v_1(\varepsilon, t)$ are analytical with respect to the parameter ε.

We will formulate the problem of the stability of the periodic solution (1.3) of system (1.1) and first consider a system of approximation, linear in ε (everywhere henceforth summation over j and s is carried out from 1 to 2)

$$\dot{x}_s = \varepsilon \sum_j a_{sj}(t) x_j + \left(b_s^0 + \varepsilon b_s(t)\right) y, \quad s = 1, 2$$

$$\dot{y} = \sum_j \left(a_j^0 + \varepsilon a_j(t)\right) x_j + \varepsilon a_{33}(t) y \tag{1.4}$$

By virtue of thee reversibility of the initial system (1.1) and the symmetry of the solution (1.3), Eqs (1.4) will also be invariant [2] with respect to each of the following transformations: (a) $(x_1, x_2, y, t) \to (x_1, x_2, -y, -t)$ and (b) $(x_1, x_2, y, t) \to (-x_1, -x_2, y, -t)$.

This means that in system (1.4) the functions $a_{sj}(t)$ are odd while $b_s(t)$ and $a_j(t)$ are even in t. Then, these functions can be represented by Fourier series in the form

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$$
\begin{aligned}
a_{sj} &= a_{sj}^{(1)} \sin t + a_{sj}^{(2)} \sin 2t + \ldots \\
b_s &= b_s^* + b_s^{(1)} \cos t + b_s^{(2)} \cos 2t + \ldots \\
a_j &= a_j^* + a_j^{(1)} \cos t + a_j^{(2)} \cos 2t + \ldots
\end{aligned}
\tag{1.5}
$$

The right-hand side of system (1.4) depends on ε. When $\varepsilon = 0$ system (1.4) is autonomous and the roots of the characteristic equation are

$$\lambda_{1,2} = \pm\sqrt{a_1^0 b_1^0 + a_2^0 b_2^0}, \quad \lambda_3 = 0$$

When $a_1^0 b_1^0 + a_2^0 b_2^0 < 0$ we have pure imaginary root $\lambda_{1,2} = \pm i\omega$, and the necessary condition for solution (1.3) to be stable is satisfied. This condition is sufficient for linear system (1.4) to be stable for small $\varepsilon \neq 0$, provided there is no parametric resonance $2\omega = p \in \mathbf{N}$ [3].

Parametric resonance, as a rule, leads to instability [3]. derived from the fact that one coefficient in the normal form of linear system (1.4) does not vanish. The calculation of this coefficient is problem related to normalizing the system, periodic in t and dependent on the parameter ε. However it is sufficient to calculate this coefficient in the first approximation in ε [3], which somewhat simplifies the problem. Below we obtain specific formulae which enable one to calculate this coefficient in the general case of third-order system (1.1).

System (1.1) always has one characteristic exponent equal to zero [2]. Hence, by the Lyapunov–Floquet theory system (1.1) allows of a first integral of the form

$$V(x_1, x_2, y, \varepsilon, t) = \sum_j A_j(\varepsilon, t) x_j + B(\varepsilon, t) y = h \quad (h = \text{const}) \tag{1.6}$$

with coefficients that are periodic in t. In this, in view of the reversibility of the initial system (1.1), the functions $A_j(\varepsilon, t)$ will be even while the function $B(\varepsilon, t)$ will be odd. Moreover, obviously, these coefficients depend on the parameter ε. We will put

$$A_j(\varepsilon, t) = \alpha_j^0 + \varepsilon\alpha_j(\varepsilon, t), \quad B(\varepsilon, t) = \varepsilon\beta(\varepsilon, t) \tag{1.7}$$

and we will calculate the total derivative of the function V by virtue of system (1.1). We have

$$\frac{dV}{dt} = \sum_j (\dot{\alpha}_j^0 + \varepsilon\dot{\alpha}_j) x_j + \varepsilon\dot{\beta} y + \sum_j (\alpha_j^0 + \varepsilon\alpha_j)\left[\sum_s \varepsilon a_{js} x_s + (b_j^0 + \varepsilon b_j) y\right] +$$
$$+\varepsilon\beta\left[\sum_s (a_s^0 + \varepsilon a_s) x_s + \varepsilon a_{33} y\right]$$

From the condition for this derivative to be equal to zero we obtain equations which satisfy the unknown functions α_j and β. For this we equate the coefficients of like powers of ε. For the coefficients of ε^0 and ε^1 we obtain

$$\dot{\alpha}_j^0 = 0, \quad j = 1, 2, \quad \sum_s \alpha_s^0 b_s^0 = 0 \tag{1.8}$$

$$\dot{\alpha}_j(0, t) + \sum_s \alpha_s^0(0, t) a_{sj} + a_j^0 \beta(0, t) = 0, \quad j = 1, 2$$

$$\dot{\beta}(0, t) + \sum_s (\alpha_s^0 b_s(t) + \alpha_s(0, t) b_s^0) = 0 \tag{1.9}$$

The first of these equations serves to determine α_j^0 and gives a certain freedom in choosing them. We will put

$$\alpha_j(0, t) = \alpha_{j0} + \alpha_{j1} \cos t + \alpha_{j2} \cos 2t + \ldots$$
$$\beta(0, t) = \beta_1 \sin t + \beta_2 \sin 2t + \ldots$$

(α_{j0}, α_{j1}, β_j are constants) and substitute these expressions into system (1.9). As a result we have

$$\alpha_{s1}\sin t+2\alpha_{s2}\sin 2t+\ldots=a_s^0(\beta_1\sin t+\beta_2\sin 2t+\ldots)+$$
$$+\sum_j \alpha_j^0 a_{js}^{(1)}\sin t+\sum_j \alpha_j^0 a_{js}^{(2)}\sin 2t+\ldots,\quad s=1,2$$
$$-(\beta_1\cos t+2\beta_2\cos 2t+\ldots)=\sum_j b_j^0\alpha_{j0}+\sum_j b_j^0\alpha_{j1}\cos t+\sum_j b_j^0\alpha_{j2}\cos 2t+$$
$$+\sum_j \alpha_j^0 b_j^*+\sum_j \alpha_j^0 b_j^{(1)}\cos t+\sum_j \alpha_j^0 b_j^{(2)}\cos 2t+\ldots$$

Then, equating coefficients of like harmonics we obtain

$$\sum_j (b_j^0\alpha_{j0}+\alpha_j^0 b_j^*)=0 \tag{1.10}$$
$$k\alpha_{sk}=a_s^0\beta_k+\sum_j \alpha_j^0 a_{js}^{(k)},\quad s=1,2,\quad -k\beta_k=\sum_j (b_j^0\alpha_{jk}+\alpha_j^0 b_j^{(k)}),\quad k=1,2,\ldots$$

From the first two equations of (1.10) we determine α_{1k} and α_{2k} and substitute them into the third equation. We then obtain a single linear equation for determining β_k

$$(\sum_j a_j^0 b_j^0+k^2)\beta_k+k\sum_j \alpha_j^0 b_j^{(k)}+\alpha_1^0\sum_j b_j^0 a_{1j}^{(k)}+a_2^0\sum_j b_j^0 a_{2j}^{(k)}=0 \tag{1.11}$$

where α_j^0 satisfy condition (1.8).

If, for any natural k, we have

$$\Delta\equiv\sum_j a_j^0 b_j^0+k^2\neq 0$$

then all β_k and, of course, also α_{jk}, are determined uniquely. This condition is always satisfied when there is no parametric resonance.

If there is parametric resonance $2\omega = p = 2l-1$, $l\in\mathbf{N}$, we have

$$\sum_j a_j^0 b_j^0=-(2l-1)^2/4$$

and $\Delta = 0$ when $(2l-1)^2 = 4k^2$, which is impossible. Consequently, for odd p the transformation is also determined uniquely.

In these cases we put

$$\alpha_1^0=b_2^0,\quad \alpha_2^0=-b_1^0,\quad \alpha_{10}=-b_2^0,\quad \alpha_{20}=b_1^0-\sum_j \alpha_j^0 b_j^*/b_2^0 \tag{1.12}$$

Then the coefficients α_{jk} and β_k are calculated from the formulae

$$\alpha_{jk}=(a_j^0 f_k+g_k)/k,\quad \beta_k=f_k, \tag{1.13}$$
$$f_k=-(\sum_j b_j^0(b_2^0 a_{1j}^{(k)}-b_1^0 a_{2j}^{(k)})+k(b_2^0 b_1^{(k)}-b_1^0 b_2^{(k)}))(\sum_j b_j^0 a_j^0+k^2)^{-1}$$
$$g_k=b_2^0 a_{11}^{(k)}-b_1^0 a_{21}^{(k)}$$

We will now assume that parametrics resonance occurs $2\omega = p = 2l$, $l\in\mathbf{N}$. Then, when $k = l$ we have $\Delta = 0$. In this case we obtain from (1.11) that the linear homogeneous from of α_j^0 is equal to zero. But α_j^0 also satisfy (1.8). Hence we obtain $\alpha_j^0 = 0$.

Hence, when $\omega = l\in\mathbf{N}$ we have in (1.7)

$$\alpha_j^0=0,\quad \sum_j b_j^0\alpha_{j0}=0,\quad \alpha_{jk}=a_j^0\beta_k/k,\quad k\in\mathbf{N}$$

while the coefficients β_k are arbitrary. In this case integral (1.6) does not enable us to eliminate one of the equations (in x_1 or x_2) from the system.

We can also use formulae (1.13) in the special case of parametric resonance $\omega = l$, when the first l harmonics are not present in system (1.4). In this case Eqs (1.11) give: $\beta_1 = \ldots = \beta_{k-1} = 0$, β_k is an arbitrary quantity and $\beta_{k+1}, \beta_{k+2}, \ldots$ are uniquely defined. In this case also formulae (1.12) remain true.

Thus, the coefficients α_{jk} and β_k in the integral are calculated from formulae (1.12) and (1.13) in the following cases:

(a) there is no parametric resonance;

(b) there is parametric resonance $2\omega = p = 2l-1, l \in \mathbf{N}$;

(c) there is parametric resonance $\omega = l \in \mathbf{N}$ but system (1.4) does not contain the first l harmonics.

In this cases we reduce the order of system (1.4) using integral V. To do this we express from (1.6)

$$x_1 = \Delta_1(V - (\alpha_2^0 + \varepsilon\alpha_2)x_2 - \varepsilon\beta y), \quad \Delta_1 = (\alpha_1^0 + \varepsilon\alpha_1)^{-1}$$

and substitute the expression obtained into system (1.4). We obtain

$$\dot{x}_2 = (-\varepsilon a_{21}(\alpha_2^0 + \varepsilon\alpha_2)\Delta_1 + \varepsilon a_{22})x_2 + (-\varepsilon^2 a_{21}\beta\Delta_1 + b_2^0 + \varepsilon b_2)y + q(t)V$$
$$\dot{y} = (-(a_1^0 + \varepsilon a_1)(\alpha_2^0 + \varepsilon\alpha_2)\Delta_1 + a_2^0 + \varepsilon a_2)x_2 + (-\varepsilon\beta(a_1^0 + \varepsilon a_1)\Delta_1 + \varepsilon a_{33})y + r(t)V$$

(the coefficients $q(t)$ and $r(t)$ are 2π-periodic functions).

In the integral manifold $V = 0$, we have the second-order system

$$\dot{x}_2 = \eta_1 x_2 + \xi_1 y, \quad \dot{y} = \xi_2 x_2 + \eta_2 y \tag{1.14}$$
$$\eta_j = \varepsilon(\eta_{j1}\sin t + \eta_{j2}\sin 2t + \ldots) + \varepsilon^2(\ ) + \ldots$$
$$\xi_j = \xi_j^0 + \varepsilon(\xi_{j0} + \xi_{j1}\cos t + \xi_{j2}\cos 2t + \ldots) + \varepsilon^2(\ ) + \ldots, \quad j = 1, 2$$

We will express the coefficients η_j, ξ_j in terms of the coefficients of system (1.4), specified by formulae (1.5). We obtain

$$\eta_1 = \varepsilon\left(a_{22} - a_{21}\frac{\alpha_2^0}{\alpha_1^0}\right) = \varepsilon\left(\left(a_{22}^{(1)} - a_{21}^{(1)}\frac{\alpha_2^0}{\alpha_1^0}\right)\sin t + \left(a_{22}^{(2)} - a_{21}^{(2)}\frac{\alpha_2^0}{\alpha_1^0}\right)\sin 2t + \ldots\right)$$
$$\eta_2 = \varepsilon\left(a_{33} - a_1^0\frac{\beta}{\alpha_1^0}\right) = \varepsilon\left(\left(a_{33}^{(1)} - \beta_1\frac{a_1^0}{\alpha_1^0}\right)\sin t + \left(a_{33}^{(2)} - \beta_2\frac{a_1^0}{\alpha_1^0}\right)\sin 2t + \ldots\right)$$

$$\xi_1 = b_2^0 + \varepsilon b_2 = b_2^0 + \varepsilon(b_2^* + b_2^{(1)}\cos t + b_2^{(2)}\cos 2t + \ldots) \tag{1.15}$$
$$\xi_2 = a_2^0 - a_1^0\frac{\alpha_2^0}{\alpha_1^0} + \varepsilon\left(a_2 - a_1\frac{\alpha_2^0}{\alpha_1^0} - a_1^0\frac{\alpha_2\alpha_1^0 - \alpha_1\alpha_2^0}{(\alpha_1^0)^2}\right) =$$
$$= a_2^0 - a_1^0\frac{\alpha_2^0}{\alpha_1^0} + \varepsilon\left(a_2^* - a_1^*\frac{\alpha_2^0}{\alpha_1^0} - a_1^0\frac{\alpha_{20}\alpha_1^0 - \alpha_{10}\alpha_2^0}{(\alpha_1^0)^2} + \right.$$
$$\left. + \left(a_2^{(1)} - a_1^{(1)}\frac{\alpha_2^0}{\alpha_1^0} - a_1^0\frac{\alpha_{21}\alpha_1^0 - \alpha_{11}\alpha_2^0}{(\alpha_1^0)^2}\right)\cos t + \ldots\right)$$

Consider the system obtained from system (1.14) averaged over the period of the coefficients

$$\dot{x}_2 = (\xi_1^0 + \varepsilon\xi_{10})y, \quad \dot{y} = (\xi_2^0 + \varepsilon\xi_{20})x_2 \tag{1.16}$$

We will denote by

$$\omega_*^2 = -(\xi_1^0 + \varepsilon\xi_{10})(\xi_2^0 + \varepsilon\xi_{20})$$

the frequency of the oscillations of the averaged system (1.16), which when $\varepsilon = 0$ is identical with the frequency ω of the oscillations of system (1.2)

$$\omega_* = \omega + \varepsilon\omega_1 + O(\varepsilon^2), \quad \omega_1 = -\frac{1}{2\sqrt{|\xi_1^0\xi_2^0|}}(\xi_{10}\xi_2^0 + \xi_{20}\xi_1^0) \tag{1.17}$$

By means of the transformation

$$x_2 = -\frac{x_2^*}{\sqrt{|\xi_2^0|}}\left(1 - \varepsilon\frac{\xi_{20}}{2|\xi_2^0|}\right), \quad y = \frac{y^*}{\sqrt{|\xi_1^0|}}\left(1 - \varepsilon\frac{\xi_{10}}{2|\xi_1^0|}\right)$$

we can reduce system (1.16) to the form

$$\dot{x}_2^* = -(\omega + \varepsilon\omega_1)y^*, \quad \dot{y}^* = (\omega + \varepsilon\omega_1)x_2^* \tag{1.18}$$

Then the corresponding system (1.14) takes the form

$$\begin{aligned}
&\dot{x}_2^* = -(\omega + \varepsilon\omega_1)y^* + \varepsilon(\eta_{11}\sin t + \ldots)x_2^* + \varepsilon r(\xi_{11}\cos t + \ldots)y^* \\
&\dot{y}^* = (\omega + \varepsilon\omega_1)x_2^* + \varepsilon r^{-1}(\xi_{21}\cos t + \ldots)x_2^* + \varepsilon(\eta_{21}\sin t + \ldots)y^* \\
&r = |\xi_2^0/\xi_1^0|^{1/2}
\end{aligned} \tag{1.19}$$

We now use the complex-conjugate variables $z = x_2^* + iy^*, \bar{z} = x_2^* - iy^*$ for system (1.19).
We consider the case of second-order resonance

$$2\omega = p, \quad p \in \mathbf{Z}$$

In this case the equation for z can be written in the form

$$\begin{aligned}
&\dot{z} = i(\omega + \varepsilon\omega_1)z + \tfrac{1}{4}i\varepsilon(z[F_- - G_-]e^{ipt} - z[F_- + G_-]e^{-ipt} + \\
&+\bar{z}[F_+ + G_+]e^{ipt} + \bar{z}[-F_+ + G_+]e^{-ipt}) \\
&F_\pm = -\eta_{1p} \pm \eta_{2p}, \quad G_\pm = r^{-1}\xi_{2p} \pm r\xi_{1p}
\end{aligned} \tag{1.20}$$

According to results obtained previously in [3], if the expansion of the coefficients in system (1.4) begins with the pth harmonic, solution (1.3) of system (1.1) will be unstable in the first approximation [3] if the following inequality is satisfied

$$\varkappa^2 > \omega_1^2, \quad \varkappa = F_+ + G_+ \tag{1.21}$$

where $\varkappa$ is the coefficient of $\bar{z}e^{ipt}$ in Eq. (1.20).
We have thereby obtained an explicit expression for the resonance coefficient $\varkappa$ for parametric resonance.

2. THE PROBLEM OF A HEAVY RIGID BODY ROLLING ALONG A STRAIGHT LINE ON A PLANE

The rolling of a heavy rigid body along an absolutely rough plane described by the system [4]

$$\begin{aligned}
&\boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{\Theta}\boldsymbol{\omega}) = -m\boldsymbol{\rho} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) - g\boldsymbol{\gamma}) \\
&\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0
\end{aligned} \tag{2.1}$$

where m is the mass of the body, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ is the vector of the instantaneous angular velocity, $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ is the unit vector at the point of contact between the body and the plane, directed vertically upwards, g is the acceleration due to gravity, $\Theta = \text{diag}\,\{A, B, C\}$ is the central tensor of inertia of the body, and $\boldsymbol{\rho} = (x, y, z)$ is the radius vector drawn from the centre of mass to the contact point. The relation between the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$ is established using the equation of the surface of the body; if this equation is written in the form $F(\boldsymbol{\rho}) = 0$, we have

$$\boldsymbol{\gamma} = -\text{grad}\, F(\boldsymbol{\rho}) \,/\, |\text{grad}\, F(\boldsymbol{\rho})| \tag{2.2}$$

We will write the equations of motion (2.1) in terms of the projections ω_1, ω_2, and ω_3 of the vector of the instantaneous angular velocity $\boldsymbol{\omega}$ onto the axes of the coupled system of coordinates and the projections of the unit vector $\boldsymbol{\gamma}$

$$\begin{aligned}
&A\dot{\omega}_1 + \omega_2\omega_3(C-B) = mg(y\gamma_3 - z\gamma_2) - m(\dot{\omega}_1 y^2 - \dot{\omega}_2 xy - \dot{\omega}_3 xz + \dot{\omega}_1 z^2) - \\
&-m(\omega_1 \dot{y}y - \omega_2 \dot{x}y - \omega_3 \dot{x}z + \omega_1 \dot{z}z) - m(\omega_1\omega_3 xy - \omega_2^2 zy + \omega_2\omega_3 y^2 - \\
&-\omega_3\omega_2 z^2 + \omega_3^2 yz - \omega_1\omega_2 xz) \\
&\dot{\gamma}_1 + \omega_2\gamma_3 - \omega_3\gamma_2 = 0 \\
&(1\,2\,3, x\,y\,z, A\,B\,C)
\end{aligned} \tag{2.3}$$

The relation between the vector $\boldsymbol{\gamma}$ and the vector $\boldsymbol{\rho}(x, y, z)$, drawn at the point of contact of the body and the plane, in the case of body bounded by the surface of an ellipsoid, is given by Eq. (2.2)

$$\gamma_1 = -\frac{x}{a^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2} \quad (1\,2\,3,\ \ x\,y\,z,\ \ a\,b\,c) \tag{2.4}$$

Hence, the problem of the motion of a heavy rigid body along an absolutely rough fixed horizontal plane is described by a closed system of sixth-order scalar differential equations in x, y, z, ω_1, ω_2, and ω_3. System (2.3) allows of first integrals – the integral of energy and the geometric integral

$$\begin{aligned}
&m[\omega_1^2(z^2 + y^2) + \omega_2^2(x^2 + z^2) + \omega_3^2(y^2 + x^2) - \\
&-2(\omega_1\omega_2 yx + \omega_2\omega_3 zy + \omega_3\omega_1 xz)] + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - \\
&-2mg(x\gamma_1 + y\gamma_2 + z\gamma_3) = 2h \;\; (h = \text{const}) \\
&\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1
\end{aligned} \tag{2.5}$$

Noted that the presence of these integrals enables us, in principle, to describe the problem by a system of fourth-order differential equations, which depend on h. However, this reduction presents a certain problem because the first of the integrals contains both the projections γ_1, γ_2 and γ_3 and the coordinates of the contact point x, y, z: the relation between these quantities is given by the fairly lengthy formulae (2.4).

System of equations (2.3) possesses the integral manifold [5] in which

$$\omega_1 = \omega_2 = 0, \quad \gamma_3 = 0 \tag{2.6}$$

while the change of variables γ_1, γ_2 and ω_3 is defined by the system

$$\begin{aligned}
&[C + m(x^2 + y^2)]\dot{\omega}_3 = m(g(\gamma_2 x + \gamma_1 y) - \omega_3(x\dot{x} + y\dot{y})) \\
&\dot{\gamma}_1 - \omega_3\gamma_2 = 0, \quad \dot{\gamma}_2 + \omega_3\gamma_1 = 0
\end{aligned} \tag{2.7}$$

In system (2.3) we will put the variables ω_1, ω_2 and γ_3 equal to zero. Then the first, second and fourth equations of the system become identities, and the system of third-order equations (2.7), defining the change in the variables γ_1, γ_2 and ω_3, remains.

This manifold corresponds to a particular motion in which the point of contact between the body and the plane describes one of the principal sections, namely, the section situated in the xGy plane.

System of equations (2.3) is invariant under the transformation

$$(t,\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2,\gamma_3)\to(-t,-\omega_1,-\omega_2,-\omega_3,\gamma_1,\gamma_2,\gamma_3)$$

We substitute the variables $-\omega_1$, $-\omega_2$, $-\omega_3$, γ_1, γ_2, γ_3 into system of equations (2.3) instead of the variables ω_1, ω_2, ω_3, γ_1, γ_2, and γ_3, and replace t by $-t$. Then the equations retain their form. Moreover, system (2.3), (2.4) is invariant under each of the transformations of the form $(t, \omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3) \to (-t, -\omega_1, \omega_2, \omega_3, -\gamma_1, \gamma_2, \gamma_3)$ taking into account the cyclic replacement of variables. This means that system (2.3) is a linearly reversible system [6] with four fixed sets. The property of reversibility enables us to use the methods developed in recent years for investigating reversible systems (see, for example, [7] to investigate system (2.6).

We rotate the system of coordinates xyz in the xGy plane around the z axis by an angle φ and we will represent the deflection of the z axis from the vertical by the angle θ. This is achieved by a transformation which relates the "old" and "new" variables by the formulae [5]

$$\begin{gathered}\gamma_1=\sin\theta\cos\varphi,\quad \gamma_2=\sin\theta\sin\varphi,\quad \gamma_3=\cos\theta(0\le\theta\le\pi)\\ \omega_1=p\cos\varphi-q\sin\varphi,\quad \omega_2=p\sin\varphi+q\cos\varphi,\quad \omega_3=r\end{gathered}\tag{2.8}$$

Obviously φ is the angle between the Gx axis and the $G\xi$ axis – the line of intersection of the xGy plane and the plane passing through the Gz axis and the vertical, and θ is the angle between the Gz axis and the vertical.

Replacement (2.8) enables us to take the geometric integral into account automatically. As a result the problem is described by five independent variables p, q, r φ and θ. Further, we eliminate r using integral and obtain a system of fourth-order differential equations

$$\begin{aligned}&\frac{dp}{dt}=S^{-1}\{[X_1+Y_1]\cos\varphi+[X_2+Y_2]\sin\varphi\}+q\dot\varphi\\ &\frac{dq}{dt}=S^{-1}\{[-X_1+Y_1]\sin\varphi+[X_2+Y_2]\cos\varphi\}-p\dot\varphi\\ &\frac{d\theta}{dt}=-q\\ &\frac{d\varphi}{dt}=-r+p\operatorname{ctg}\theta\end{aligned}\tag{2.9}$$

where

$$S=AB+Am(x^2+z^2)+Bm(y^2+z^2)+m^2z^2(x^2+y^2+z^2)$$

$$X_1=(B+m(x^2+z^2))(m\dot\omega_3xz+X),\quad Y_1=mxy(m\dot\omega_3yz+Y)$$

$$X_2=mxy(m\dot\omega_3xz+X),\quad Y_2=(A+m(y^2+z^2))(m\dot\omega_3yz+Y)$$

$$\begin{aligned}X=(B-C)\omega_2\omega_3+m\{&g(\gamma_3y-\gamma_2z)-\omega_1(x\dot x+y\dot y+z\dot z)+\\ &+\dot x(\omega_1x+\omega_2y+\omega_3z)-\omega_3y(\omega_1x+\omega_2y)+\\ &+\omega_2z(\omega_3z+\omega_1x)+yz(\omega_2^2-\omega_3^2)\}\end{aligned}$$

$$\begin{aligned}Y=(C-A)\omega_3\omega_1+m\{&g(\gamma_1z-\gamma_3x)-\omega_2(x\dot x+y\dot y+z\dot z)+\\ &+\dot y(\omega_1x+\omega_2y+\omega_3z)-\omega_1z(\omega_2y+\omega_3z)+\\ &+\omega_3x(\omega_1x+\omega_2y)+zx(\omega_3^2-\omega_1^2)\}\end{aligned}$$

(the expressions of ω_1, ω_2, γ_1, γ_2, and γ_3 in terms of p, q, θ and φ are given by formulae (2.8), and ω_3 is eliminated by the energy integral).

3. A BODY ROLLING ALONG A STRAIGHT LINE

In system of equations (2.9) the manifold (2.6) is defined by the condition

$$\theta = \pi/2, \quad p = q = 0, \quad r = \omega_3 = -\dot{\varphi} \tag{3.1}$$

where the dependence of φ on t is given by the first of Eqs (2.7)

$$\left(C + m\left(x^{*2} + y^{*2}\right)\right)\ddot{\varphi} + m\left(g\left(\gamma_2^* x^* - \gamma_1^* y^*\right) - r\left(x^* \dot{x}^* + y^* \dot{y}^*\right)\right) = 0 \tag{3.2}$$

Here and below the asterisk denotes that the value of the function is calculated on the motion (3.1) investigated.

For a specified body surface the coordinates of the point of contact between the body and the plane will be functions of the angle φ.

Below we will consider a hollow ellipsoid, i.e. a body bounded by the surface of an ellipsoid with semi-axes a, b and c and a cavity in the form of a coaxial homothetic ellipsoid with semi-axes $a(1-d)$, $b(1-d)$ and $c(1-d)$, where d is dimensionless parameter. The moments of inertia of the ellipsoid are

$$A = \lambda m\left(b^2 + c^2\right)/5, \quad \lambda = \left(1-(1-d)^5\right)\left(1-(1-d)^3\right)^{-1}$$
$$(ABC,\ abc)$$

We obtain the coordinates of the point of contact x^*, y^* and the velocities $\dot{x}^*, \dot{y}^*$

$$x^* = -\cos\varphi\Phi^{-1}(\varphi), \quad y^* = -\alpha\sin\varphi\Phi^{-1}(\varphi)$$
$$\dot{x}^* = \alpha\sin\varphi\dot{\varphi}\Phi^{-3}(\varphi), \quad \dot{y}^* = -\alpha\cos\varphi\dot{\varphi}\Phi^{-3}(\varphi)$$
$$\Phi(\varphi) = \left(\cos^2\varphi + \alpha\sin^2\varphi\right)^{1/2}, \quad \alpha = b^2/a^2$$

After substituting the expressions obtained into (3.2) we obtain a conservative system with one degree of freedom. A complete investigation of this system was carried out in [8] using the phase-plane method.

We can distinguish several forms of motion depending on the value of the constant energy h. To classify these it is convenient to use the integral in dimensionless form.

We will introduce the following characteristic quantities: l is the major axis of the ellipsoid and $\tau = \sqrt{g/l}t$ is dimensionless time. We then obtain

$$\frac{1}{2}\left[\frac{d\varphi}{d\tau}\right]^2\left(C^* + \left(\cos^2\varphi + \alpha^2\sin^2\varphi\right)\Phi^{-2}(\varphi)\right) + \Phi(\varphi) = h^*; \; h^* = \frac{h}{mgl}, \; C^* = \frac{C}{ml^2} \tag{3.3}$$

We can distinguish the following cases [8] depending on the value of the dimensionless constant h^* (or h):

(1) $h^* < \sqrt{\alpha}$ (or $h < mgb$) – motion is impossible;

(2) $\sqrt{\alpha} < h^* < 1$ (or $mgb < h < mga$) – the ellipsoid undergoes "rolling" in which the track of the point or contact on the reference plane is a segment. In these motions the value of angle φ lies in the range $(0, T)$. The angular velocity $d\varphi/dt$ is a periodic function of time, which becomes zero twice per period of the rolling.

(3) $h^* > 1$ – the ellipsoid rolls along the straight line in one direction with angular velocity $d\varphi/dt$ which is periodic in time.

Below we will investigate the problem of the stability of the rollings of the ellipsoid in one direction (case 3). Since in this case the angular velocity of the rolling of the ellipsoid retains its sign, to describe these rollings, and motions close to it, we can change to a new independent variable – the angle φ. To do this we divide the equations of system (2.9) by $d\varphi/dt$ and we obtain a 2π-periodic third-order system

$$\frac{dp}{d\varphi} = \frac{\left[X_1 + Y_1\right]\cos\varphi + \left[X_2 + Y_2\right]\sin\varphi}{S\dot{\varphi}} + q$$
$$\frac{dq}{d\varphi} = \frac{\left[-X_1 + Y_1\right]\sin\varphi + \left[X_2 + Y_2\right]\cos\varphi}{S\dot{\varphi}} - p \tag{3.4}$$

$$\frac{d\theta}{d\varphi} = -\frac{1}{S\dot{\varphi}}$$

The angular velocity $\dot{\varphi}$ on the manifold (3.1) is found from the energy integral (3.3)

$$\frac{d\varphi}{dt} = \pm\sqrt{\frac{2\left(h^* + mg\left(x^* \cos\varphi + y^* \sin\varphi\right)\right)}{\left(m\left(x^{*2} + y^{*2}\right) + C\right)}} \tag{3.5}$$

and in the general case – from the energy integral (2.5).

Without loss of generality we will consider the case for which $\dot{\varphi} > 0$, and the problem of the stability of the rolling is correctly reduced to investigating the stability of the particular solution

$$p = q = 0, \quad \theta = \pi/2 \tag{3.6}$$

of system (3.4).

4. THE EQUATIONS OF PERTURBED MOTION

We will write the system of equations in variations for system of equations (3.4) in the neighbourhood of the solution (3.6) investigated

$$\begin{aligned}
&\frac{d}{d\varphi}\delta p = \left[\frac{\cos\varphi}{S^*\dot{\varphi}}d_1 + \frac{\sin\varphi}{S^*\dot{\varphi}}d_2\right]\delta p + \left[\frac{\cos\varphi}{S^*\dot{\varphi}}d_3 + \frac{\sin\varphi}{S^*\dot{\varphi}}d_4 + 1\right]\delta q + \\
&+\frac{m}{S^*\dot{\varphi}}[\cos\varphi c_1 + \sin\varphi c_2]\delta\theta \\
&\frac{d}{d\varphi}\delta q = \left[-\frac{\sin\varphi}{S^*\dot{\varphi}}d_1 + \frac{\cos\varphi}{S^*\dot{\varphi}}d_2 - 1\right]\delta p + \left[-\frac{\sin\varphi}{S^*\dot{\varphi}}d_3 + \frac{\cos\varphi}{S^*\dot{\varphi}}d_4\right]\delta q + \\
&+\frac{m}{S^*\dot{\varphi}}[-c_1\sin\varphi + c_2\cos\varphi]\delta\theta \\
&\frac{d}{d\varphi}\delta\theta = -\frac{1}{\dot{\varphi}}\delta q
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&S^* = AB + m\left(Ax^{*2} + By^{*2}\right) \\
&d_1 = B_1a_1 + mx^*y^*a_2, \quad d_2 = mx^*y^*a_1 + A_1a_2 \\
&d_3 = B_1a_3 + mx^*y^*a_4, \quad d_4 = mx^*y^*a_3 + A_1a_4 \\
&B_1 = B + mx^{*2}, \quad A_1 = A + my^{*2} \\
&a_1 = \omega_3\left((B-C)\sin\varphi - my^*X_+\right) + my^*X_- \\
&a_2 = \omega_3\left((C-A)\cos\varphi + mx^*X_+\right) - mx^*X_- \\
&a_3 = \omega_3\left((B-C)\cos\varphi + my^*X_-\right) \\
&a_4 = \omega_3\left(-(C-A)\sin\varphi - mx^*X_-\right) \\
&X_+ = x^*\cos\varphi + y^*\sin\varphi, \quad X_- = \dot{x}^*\sin\varphi - \dot{y}^*\cos\varphi
\end{aligned} \tag{4.2}$$

$$c_1 = \frac{\dot{\omega}_3 c^2 x^*}{\Delta_2}\left(B + m\left(x^{*2} + y^{*2}\right)\right) + B_1 b_1 + m x^* y^* b_2$$

$$c_2 = \frac{\dot{\omega}_3 c^2 y}{\Delta_2}\left(A + m\left(x^{*2} + y^{*2}\right)\right) + m x^* y^* b_1 + A_1 b_2$$

$$b_1 = \frac{c^2}{\Delta_2}\left(\dot{x}^* \omega_3 - \omega_3^2 y^* - g \sin\varphi\right) - g y^*$$

$$b_2 = \frac{c^2}{\Delta_2}\left(\dot{y}^* \omega_3 + \omega_3^2 x^* + g \cos\varphi\right) + g x^*$$

$$\Delta_2 = \left(a^2 \cos^2\varphi + b^2 \sin^2\varphi\right)^{1/2}$$

Note that in the system obtained the relation $\omega_3 = r = -\dot{\varphi}$ is specified by expression (3.5) while $\dot{\omega}_3 = -\ddot{\varphi}$. In addition $\dot{x}^* \cos\varphi + \dot{y}^* \sin\varphi = 0$.

As a result we obtain a linear reversible system with periodic coefficients, invariant under the transformation $(\varphi, \delta p, \delta q, \delta\theta) \to (-\varphi, \delta p, -\delta q, \delta\theta)$.

We will reduce system (4.1) to dimensionless form. To do this, using the quality l and the dimensionless time τ introduced earlier, we introduce the following dimensionless variables

$$p_1 = \sqrt{\frac{l}{g}} p, \quad q_1 = \sqrt{\frac{l}{g}} q, \quad r_1 = \omega_3^* = \sqrt{\frac{l}{g}} \omega_3; \quad \omega_3^{*\prime} = \frac{d\omega_3^*}{d\tau} = \frac{l}{g}\frac{d\omega_3}{dt}$$

As a result, in the new variables we obtain a dimensionless system of equations in variations.

5. THE ROLLING OF AN ELLIPSOID CLOSE TO AN ELLIPSOID OF REVOLUTION

We will investigate the rolling of an ellipsoid which is close to an ellipsoid of revolution. In this case we use $b = a\sqrt{1 + \varepsilon}$ (ε is small parameter).

When $\varepsilon = 0$ we have an ellipsoid of revolution, which rolls with constant angular velocity $d\varphi/d\tau$, defined by relation (3.5)

$$\frac{d\varphi}{d\tau} = \sqrt{\frac{10\left(h^* - 1\right)}{2\lambda + 5}} = \text{const}$$

The system of equations in variations (4.1) in the case of an ellipsoid of revolution has the form

$$\frac{d}{d\varphi}\delta p = \frac{2}{\beta + 1}\delta q$$

$$\frac{d}{d\varphi}\delta q = -\frac{\xi}{\eta}\delta p + \frac{5(\beta - 1)\sqrt{\xi}}{\eta g}\delta\theta \tag{5.1}$$

$$\frac{d}{d\varphi}\delta\theta = -\frac{\sqrt{\xi}}{g}\delta q$$

$$\xi = 2\lambda + 5, \quad \eta = \lambda(\beta + 1) + 5, \quad g = \sqrt{10\left(h^* - 1\right)}, \quad \beta = c^2 / a^2$$

We will calculate the roots of the characteristic equation of this system

$$\lambda_{1,2} = \pm\sqrt{-\frac{\xi}{\eta}\left(\frac{\beta - 1}{2\left(h^* - 1\right)} + \frac{2}{\beta + 1}\right)}, \quad \lambda_3 = 0$$

Hence it can be seen that if the following condition of satisfied

$$\beta^2 > 5 - 4h^* \tag{5.2}$$

the roots $\lambda = \pm i\omega$ of the characteristic equation will be pure imaginary.

Note that condition (5.2) is the necessary and sufficient condition for the Lyapunov stability [9] of the rolling of the ellipsoid of revolution.

We will now consider the rolling of an ellipsoid, close to an ellipsoid of revolution ($\varepsilon \neq 0$). In this case the coefficients of system (4.1), defined by formulae (4.2), will depend on ε.

We will denote the variations δ, δq and $\delta\theta$ by x_1, y and x_2. System (4.1) then takes the form (1.4) with the coefficients

$$\begin{aligned}
&a_{11}(\varphi) = \frac{\beta\xi}{2(\beta+1)\eta}\sin 2\varphi \\
&a_{12}(\varphi) = \frac{30\lambda(1-\beta) - 25(\beta+1) + 8\lambda^2 + 20h^*\eta}{\sqrt{h^*-1}(2\lambda+5)^{3/2}(\lambda+5+\beta(\eta-5/2))}\sin 2\varphi \\
&a_{21}(\varphi) = 0, \quad a_{22}(\varphi) = 0, \quad a_{33}(\varphi) = -\frac{\beta(\lambda+5)}{(\beta+1)\eta}\sin 2\varphi \\
&b_1^0 = \frac{2}{\beta+1}, \quad b_2^0 = -\frac{\sqrt{\xi}}{g} \\
&b_1(\varphi) = \frac{\beta}{(\beta+1)^2} + \frac{\beta}{(\beta+1)^2}\cos 2\varphi \\
&b_2(\varphi) = \frac{(\lambda+5/2)(1-2h^*)}{4(h^*-1)g\sqrt{\xi}} + \frac{(\lambda-5/2+5h^*)}{g\sqrt{\xi}}\cos 2\varphi \\
&a_1^0 = -\frac{\xi}{\eta}, \quad a_2^0 = \frac{5(\beta-1)\sqrt{\xi}}{\eta g} \\
&a_1(\varphi) = -\frac{\lambda\beta(\lambda+5/2)}{\eta^2} + \frac{(\lambda^2\beta + 15/2\lambda\beta + 5\lambda + 25)}{\eta^2}\cos 2\varphi \\
&a_2(\varphi) = -\frac{5\sqrt{\xi}}{4\eta^2 g}\left(\lambda\beta^2 + (4\lambda+15)\beta - \lambda - 5 + \frac{(5(1-2h^*)(\beta-1))\sqrt{\xi}}{g^2}\right) + \\
&+\left(\frac{5\beta g}{\eta\sqrt{\xi}} + \frac{5/4(\lambda\beta^2 + (4\lambda+15)\beta - \lambda - 5)\sqrt{\xi}}{\eta^2 g} + \frac{25/4(1-\beta)(2\lambda - 5 + 10h^*)}{\eta^2\xi^{1/2}g^3}\right)\cos 2\varphi
\end{aligned} \tag{5.3}$$

According to well-known results [3], the characteristic exponents of the system are identical with the characteristic exponents $\pm i\omega_*$ of the average system apart from terms of the order of ε^2 and will be pure imaginary when condition (5.2) is satisfied and when there are no second-order resonances ($2\omega = p, p \in \mathbf{Z}$).

It follows from formulae (5.3) that system (1.4) only contains an even harmonic. Consequently, only the resonance $\omega = 1$ ($p = 2$) is possible.

In explicit form, the resonance relation is

$$\frac{\sqrt{\xi}}{\eta}\left(\frac{\beta-1}{2(h^*-1)} + \frac{2}{\beta+1}\right) = 1 \tag{5.4}$$

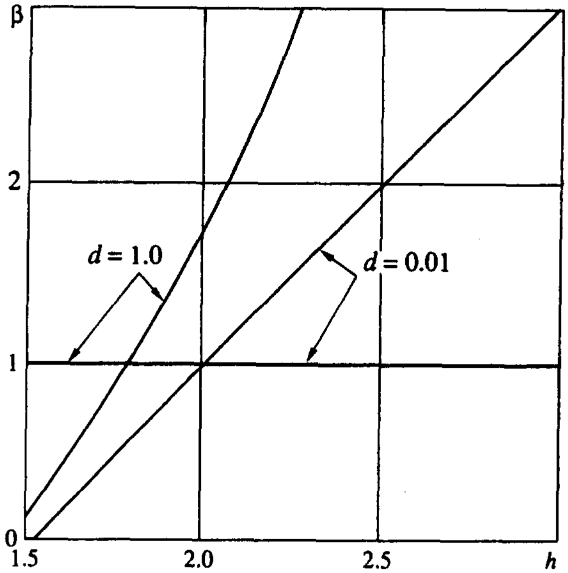

Fig. 1

Obviously this relation is satisfied for any h^* and $\beta = 1$ (the case of a sphere). Consequently, for an ellipsoid, close to a sphere, rolling always occurs when there is parametric resonance. In Fig. 1 the horizontal straight line corresponds to a sphere.

In the case when $\beta \neq 1$ relation (5.4) gives one other $\beta = \beta(h^*)$ curve, corresponding to resonance. Hence it follows that each ellipsoid, close to an ellipsoid of revolution, has its own "individual" resonance angular velocity (the quantity h^*).

In Fig. 1 the curves of $\beta = \beta(h^*)$ are given for $d = 0.01$ (a thin-walled ellipsoid) and $d = 1.0$ (a uniform ellipsoid). The masses of the ellipsoids are the same.

Further investigations were made numerically. We checked the instability condition $\varkappa^2 > \omega_1^2$ on resonance curves (5.4).

Theorem 1. Parametric resonance leads to instability of the rolling of an ellipsoid close to an ellipsoid of revolution.

Hence it follows, in particular, that the rolling of a dynamically symmetrical ellipsoid, close to a sphere, in the case when the axis of symmetry does not coincide with the horizontal axis, is unstable.

6. RESULTS AND CONCLUSIONS ON STABILITY

We will now calculate the characteristic exponents in the case of an arbitrary ellipsoid.

The dimensionless system of equations in variations with coefficients that are 2π=periodic in φ is invariant under each of the following replacements: a) $(\varphi, \delta p_1, \delta q_1, \delta\theta) \to (-\varphi, \delta p_1, \delta q_1, -\delta\theta)$ b) $(\varphi, \delta p_1, \delta q_1, \delta\theta) \to (-\varphi, \delta p_1, \delta q_1, \delta\theta)$, i.e. it is a linear reversible system. The characteristic exponents for this system can be found by the well-known method described in [2].

We will construct a solution of the Cauchy problem of a system of equations in variations in the section $[0, 2\pi]$ with the following initial data

$$\delta p_1(0) = 0, \quad \delta\theta(0) = 0, \quad \delta q_1(0) = 1$$

Then, if $|\delta q_1(2\pi)| < 1$, then characteristic exponents of the system of equations in variations will be pure imaginary [2].

The results of numerical investigations are shown in Fig. 2 and are compared for a hollow ellipsoid and a uniform ellipsoid. Rotation occurs the c axis.

We have plotted the value of the parameter $\alpha = b^2/a^2$ along the abscissa axis and the parameter $\beta = c^2/a^2$ along the ordinate axis. It is assumed that $a > b$ $(0 \leqslant \alpha \leqslant 1)$. The regions where the necessary stability conditions are satisfied are shown hatched, sloping to the left for a uniform ellipsoid ($d = 1.0$) and sloping to the right for a hollow ellipsoid ($d = 0.01$). In each part of the figure we compare the results for the same fixed value of the angular velocity (the parameter h^*).

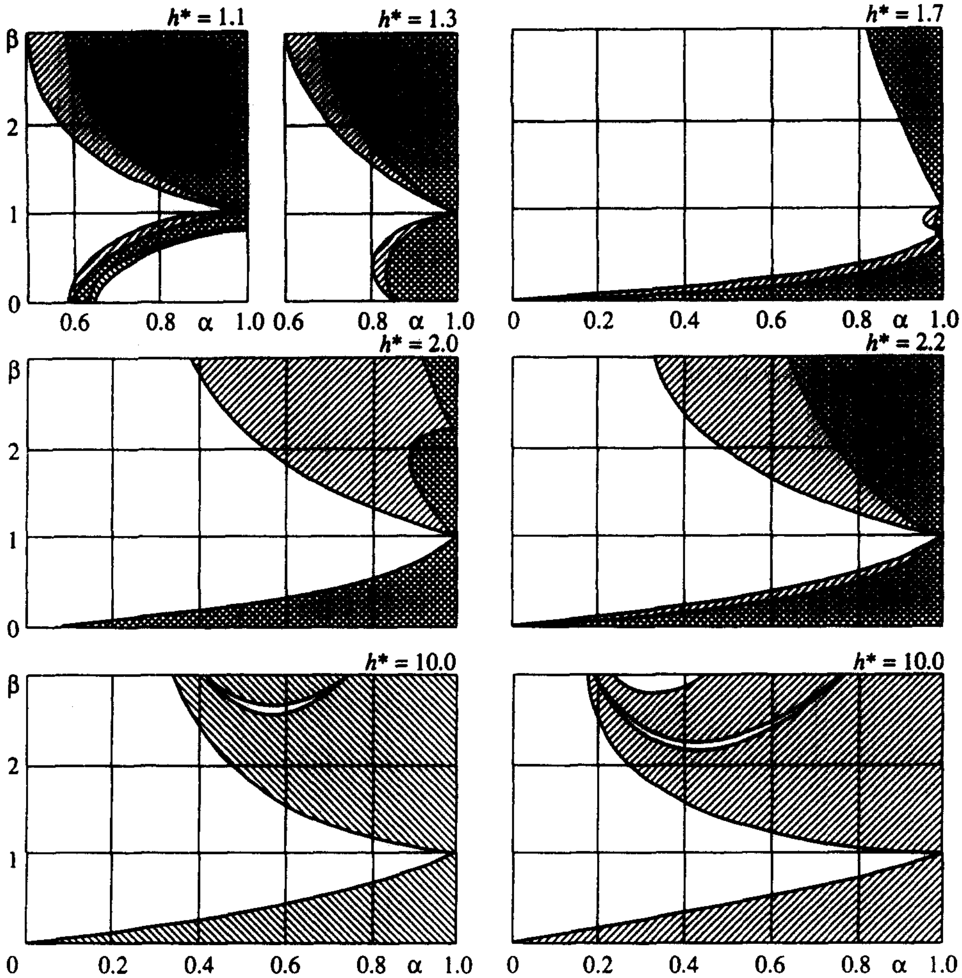

Fig. 2

The evolution of the stability regions as a function of the change in the angular velocity of the rolling can be tracked quite well in all parts of Fig. 2.

We can draw the following conclusions from an analysis of the results.

Theorem 2. Rolling of an arbitrary ellipsoid around the mean axis ($\alpha < 1$, $\beta < 1$, $\beta > \alpha$) is always unstable; rolling of the ellipsoid around the smallest axis ($\alpha < 1$, $\beta < 1$, $\beta < \alpha$) becomes stable as the angular velocity increases; the region where the necessary stability conditions for the rolling of the ellipsoid around the greatest axis are satisfied is greater the angular velocity. In this case the rolling of a hollow ellipsoid around the greatest axis is more stable than the rolling of a uniform ellipsoid around the greatest axis.

The rolling of an ellipsoid of revolution ($\alpha = 1$) around the greatest axis ($\beta > 1$) is stables; the rolling around the smallest axis ($\beta < 1$) is stable if the angular velocity is sufficiently great. These results agree completely with the conclusions reached previously in [9].

It can also be clearly seen from the above results that instability regions adjoin points of parametric resonance ($\alpha = 1$, $\beta = 1$ or $\alpha = 1$, $\beta = \beta(h^*)$) to a first approximation. Nevertheless, the conclusions regarding the stability for parametric resonance agree with the results obtained in the previous section.

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REFERENCES

1. TKHAI, V. N., Non-linear oscillations of reversible systems. *Prikl. Mat. Mekh.*, 1995, **59**, 1, 38–50.
2. TKHAI, V. N., The stability of regular Grioli precessions. *Prikl. Mat. Mekh.*, 2000, **64**, 5, 848–857.
3. TKHAI, V. N., Some problems on the stability of a reversible system with a small parameter. *Prikl. Mat. Mekh.*, 1994, **58**, 1, 3–12.
4. KARAPETYAN, A. V., Hopf bifurcation in the problem of the motion of a heavy rigid body on a rough plane. *Izv. Akad. Nauk SSSR. MTT*, 1985, **2**, 19–24.
5. TKHAI, V. N., The stability of a heavy ellipsoid of revolution rolling on a rough plane. *Izv. Ross. Akad. Nauk. MTT*, 1996, **1**, 11–16.
6. TKHAI, V. N., Reversible mechanical systems. In *The Second Polyakhov Lectures. Collected Papers*. Izd. Nauch-Issled. Inst. Khimii, St Petersburg Univ., 2000, 115–117.
7. TKHAI, V. N., Periodic motions of a system, close to an autonomous reversible system. *Prikl. Mat. Mekh.*, 2001, 65, 4, 6611–680
8. MARKEYEV, A. P., *The Dynamics of a Body in Contact with a Solid Surface*. Nauka, Moscow, 1992.
9. MINDLIN, I. M. and POZHARITSKII, G. K., The stability of the steady motions of a heavy solid of revolution on an absolutely rough horizontal plane. *Prikl. Mat. Mekh.*, 1965, **29**, 4, 742–745.

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